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# Internal symmetry in the multifractal spectrum of fully developed turbulence

D. Queiros-Conde*
*Ecole des Mines de Paris, Centre d'Energétique, 60, Bd St-Michel, 75272 Paris cedex 06, France and Groupe Turbulence, Territoires et Géométrie, 10, Allée du Clos-St-Francois, 33170 Gradignan, France*
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In the context of multifractal theory and She-Lévêque's model describing the statistics of intermittency in fully developed turbulence, we show that the multifractal dimensions can be simply written $F(\alpha)=1+\alpha^*-\alpha^*\ln(\alpha^*/2)$ with $\alpha^*=(2\beta-1-\alpha)/\ln\beta=2\beta^p$, where $p$ is the order associated to the moment $\langle\varepsilon_r^p\rangle$ (with $p\geqslant 0$) based on the rate of energy dissipation $\varepsilon_r$ and $\beta=[(1+3/\sqrt{8})^{1/3}+(1-3/\sqrt{8})^{1/3}]^3\approx 0.68$. Introducing the fractal dimensions $\Delta_p=F(\alpha)+\alpha^*\ln(\alpha^*/2)$, this leads to the recursive relation $\beta=(\Delta_{p+1}-\Delta_\infty)/(\Delta_p-\Delta_\infty)$ with $\Delta_\infty=1$. This suggests the existence of an internal symmetry in the multifractal spectrum of fully developed turbulence, which reduces considerably the number of parameters necessary to characterize intermittency statistics.

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The attempts to understand fully developed turbulence and, more particularly, the phenomenon of intermittency contain an implicit geometrical assumption for the field where energy dissipation occurs: The Kolmogorov approach (which does not take into account intermittency) assumes an homogeneous (space-filling) repartition of energy dissipation [1,2], the $\beta$-model [3] makes the hypothesis that this support is fractal [4] and, finally, the multifractal theory introduces a set of fractal structures of dimension $F(\alpha)$ which are assumed to be intimately interwined leading to the multifractal spectrum [5] with some developments brought by the idea of universal multifractals [6]. A few years ago, the She-Lévêque's model [7] (hereafter SL) has shown a great ability to describe experimental and numerical results concerning intermittency. Moreover, some theoretical works evidenced that SL model corresponds to a log-Poisson statistics for energy dissipation fluctuations [8]. The aim of our Rapid Communication is to show that if we apply the multifractal framework using the She-Lévêque's description, then it leads to the existence of an internal symmetry in the multifractal spectrum linking all the dimensions $F(\alpha)$ to each other.

Fully developed turbulent flows are usually studied by the structure functions $\langle\delta V_r{}^p\rangle$ or $\langle\varepsilon_r{}^p\rangle$, where $\delta V_r$ is the velocity difference across a distance $r$ and $\varepsilon_r$ the rate of energy dissipation per unit mass averaged over a ball of size $r$. Kolmogorov [1] postulates the existence of universal scaling laws when the scale $r$ belongs to the inertial range: $\langle\delta V_r{}^p\rangle\sim r^{\zeta_p},\langle\varepsilon_r{}^p\rangle\sim r^{\tau_p}$ (with $\zeta_p=\tau_{p/3}+p/3$ given by the refined similarity hypothesis [9]). She-Lévêque's model [7] leads to the following scaling exponents: $\zeta_p=[(2\beta-1)/3]p+2(1-\beta^{p/3})$ and $\tau_p=2(\beta-1)p+2(1-\beta^p)$ with $\beta=2/3$.

We focus here on the scaling exponent $\tau_p$ characterizing the energy dissipation statistics. Let us assume the validity of the multifractal framework and the She-Lévêque's laws. We can calculate the multifractal spectrum associated to the expression $\tau_p=2(\beta-1)p+2(1-\beta^p)$. In multifractal theory, the local scaling exponents $\alpha$ characterizing singularities in the energy dissipation field ($\varepsilon_r\sim r^{\alpha-1},r\to 0$) and the dimensions $F(\alpha)$ are linked to each other by Legendre transforms with $F(\alpha)=\min_p(p(\alpha-1)+d-\tau_p)$ where $d$ is the embedding dimension ($d=3$ for a three-dimensional turbulence). The resulting condition on the derivative is $\partial(p(\alpha-1)+d-\tau_p)/\partial p=0$, which leads to $p=\ln((2\beta-1-\alpha)/2\ln\beta)/\ln\beta$. Let us introduce the quantity $\alpha^*=(2\beta-1-\alpha)/\ln\beta=2\beta^p$; then it can be easily written

$$F(\alpha)=1+\alpha^*-\alpha^*\ln(\alpha^*/2). \tag{1}$$

We thus obtain a remarkably simple expression for $F(\alpha)$. Let us analyze some properties of the quantity $\Delta(\alpha)=F(\alpha)+\alpha^*\ln(\alpha^*/2)$. Introducing $\lambda=(2\beta-1+\ln\beta)/\ln\beta$, it is easily shown that $\Delta(\alpha)=\lambda-\alpha/\ln\beta$. This linear form means that, for every couple $(\alpha_1,\alpha_2)$, we have

$$\frac{\Delta(\alpha_1)-\lambda}{\Delta(\alpha_2)-\lambda}=\frac{\alpha_1}{\alpha_2}. \tag{2}$$

This relation implies that if, one measurement given by a couple $[\alpha_m,\Delta(\alpha_m)]$ is known, then using Eq. (2) all the spectrum of $\Delta(\alpha)$ dimensions can be deduced and, consequently, the multifractal spectrum given by $F(\alpha)$ dimensions. So, due to this internal symmetry, the number of measurements necessary to characterize a multifractal structure is considerably reduced.

Let us propose now an interpretation of this symmetry. In fact, the local exponent $\alpha$ is directly associated with the order $p$ through $p=\ln((2\beta-1-\alpha)/2\ln\beta)/\ln\beta$; let the exponent $\alpha'$ be associated with the order $p+1$. It is easily shown that $[\Delta(\alpha')-1]/[\Delta(\alpha)-1]=\beta$. If $\alpha_{\min}$ is the minimum value of $\alpha$ (corresponding to $p\to\infty$), we have $F(\alpha_{\min})=\Delta(\alpha_{\min})=1$. We thus can write $[\Delta(\alpha')-\Delta(\alpha_{\min})]/[\Delta(\alpha)-\Delta(\alpha_{\min})]=\beta$. Because there is a direct relation between $\alpha$ and $p$, the latter expression can be written

$$\frac{\Delta_{p+1}-\Delta_\infty}{\Delta_p-\Delta_\infty}=\beta. \tag{3}$$

We then observe that, in the framework of She-Lévêque's theory and multifractal approach, the calculation of the multifractal spectrum exhibits characteristic dimensions $\Delta_p$,

*Email address: queiros@cenerg.ensmp.fr

 

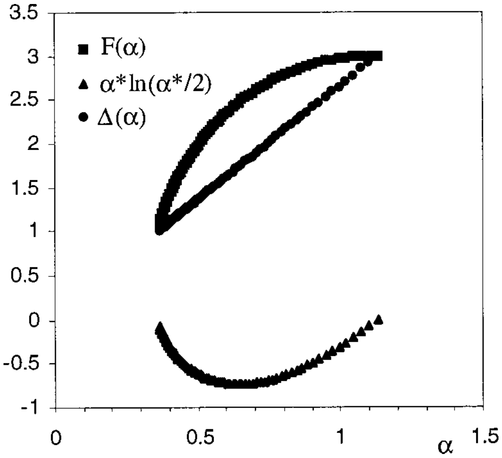


FIG. 1. Calculated values of dimensions $F(\alpha)$, $\alpha^*\ln(\alpha^*/2)$, and $\Delta(\alpha)$ represented as a function of $\alpha$ (corresponding to $p \geqslant 0$).

which are recursively linked to each other. A similar relation is obtained for any arbitrary increment $dp$ to order $p$. It is shown that, in this case, $(\Delta_{p+dp}-\Delta_\infty)/(\Delta_p-\Delta_\infty)=\beta^{dp}$. Let us remark that Eq. (3) leads immediately to the following expression: $\Delta_p=\Delta_\infty+(d-\Delta_\infty)\beta^p$. In Fig. 1, the functions $F(\alpha)$, $\alpha^*\ln(\alpha^*/2)$ and their sum $\Delta(\alpha)$ are represented, for $\alpha$ values corresponding to $p \geqslant 0$, showing the linear behavior of $\Delta(\alpha)$. Again, the interest of relation (3) through the introduction of dimensions $\Delta_p$ lies in the fact that the knowledge of one single dimension $\Delta_p$ [or equivalently $F(\alpha)$] would be enough to obtain all the other multifractal dimensions (but here for $p \geqslant 0$) having, of course, the value of the parameter $\beta$. This one will be determined in the paper, after the proposal of a slight correction to the SL model which allows its determination. For this, we need to interpret physically the dimensions $\Delta_p$.

In this interpretation of dimensions $\Delta_p$, the case $p=0$ is immediate and gives $\Delta_0=d$. When $p\rightarrow\infty$, the quantity $\Delta_\infty$ characterizes the most intermittent events occurring on filaments having a dimension $\Delta_\infty=1$. These have been observed experimentally [10] and numerically [11]. In order to proceed with the interpretation of $\Delta_p$ dimensions, we would first like to bring to She-Lévêque's model a slight correction which will allow us to calculate the number $\beta$ instead of simply assuming the value 2/3, as is made in the model [7]. SL theory defines a set of moment ratios $\varepsilon_r^{(p)}=\langle\varepsilon_r^{p+1}\rangle/\langle\varepsilon_r^p\rangle$ linked by a recursive relation $\varepsilon_r^{(p+1)}/\varepsilon_r^{(\infty)}=A_p(\varepsilon_r^{(p)}/\varepsilon_r^{(\infty)})^\beta$, where $0<\beta<1$ and $A_p$ is a constant which appears experimentally to be independent on the order $p$: $A_p=A$ [12]. $\varepsilon_r^{(\infty)}$ is the moment ratio corresponding to the filamentary fluid structures assumed to present the scaling behavior: $\varepsilon_r^{(\infty)}\sim r^{\lambda_\infty}$. To determine the parameter $\lambda_\infty$, the authors assume the scaling $\varepsilon_r^{(\infty)}\sim\delta E^\infty/t_r$, where $\delta E^\infty$ is a kinetic energy and $t_r$ a characteristic time based on the hypothesis that the mixing due to these structures having this characteristic time are homogeneous. It means that the Kolmogorov description can be applied: It gives $t_r\sim\varepsilon^{-1/3}r^{2/3}$ and then $\lambda_\infty=-2/3$. A modification of the last calculation will allow us to derive the number $\beta$ without this assumption. We know that the time $t_r$ can be estimated by $r/\langle|\delta V_r|\rangle$, with $\langle|\delta V_r|\rangle\sim r^{\zeta 1}$. The assumption $t_r\sim\varepsilon^{-1/3}r^{2/3}$ means in fact that $\zeta_1=1/3$. Instead of fixing $\zeta_1$, let us rather keep the relation $\langle|\delta V_r|\rangle\sim r^{\zeta 1}$, which gives $\lambda_\infty=\zeta_1-1$. Of course, for $\zeta_1=1/3$, we recover $\lambda_\infty=-2/3$ given by SL argument. Writing $\zeta_1$ with the general expression of $\zeta_p$ and using the relations $\lambda_\infty=\zeta_1-1$ and $\lambda_\infty=C_0(\beta-1)$ where $C_0$ is the codimension associated with the most intermittent structures (the latter relation derives from the exact result $\tau_1=0$ [2]), it is then possible to obtain an equation for $\beta$, $3\beta^{1/3}+2\beta-4=0$, whose solution is $\beta=[(1+3/\sqrt{8})^{1/3}+(1-3/\sqrt{8})^{1/3}]^3\approx0.68$. Let us notice that in this previous calculation, the choice of estimating the time $t_r$ by the quantity $r/\langle|\delta V_r|\rangle$ instead of another (such as $r/\langle|\delta V_r|^p\rangle^{1/p}$ with $p=2$, for example) is given by the implicit hypothesis to take the moment $\langle|\delta V_r|^p\rangle$, which presents the smallest level of intermittency. Knowing that intermittency increases with $p$, the order one moment $\langle|\delta V_r|\rangle$ is the most appropriate to catch a Kolmogorov's mean behavior. The slight difference of our $\beta$ value compared to the SL value 2/3 is qualitatively important. Let us recall that Chavarria *et al.* [12], trying to verify experimentally SL description, have found $\beta$=0.68 $\pm$0.03 but, due to experimental uncertainties, it is difficult to discriminate between the two values. However, we see that, using relation (3) for $p=0$, we can obtain $(\Delta_1-\Delta_\infty)/(d-\Delta_\infty)$; this implies $\Delta_1=1+2\beta\approx2.36$ (instead of 2.33 with the SL parameter). This value 2.36 is very close to the classical fractal dimension $D_f$ found in turbulence for a great number of interfaces (jets, mixing layers, and clouds), suggesting to some authors the existence of a universal behavior in turbulence [13].

Let us recapitulate our interpretation of $\Delta_p$ dimensions when $p=0$, we have $\Delta_0=d$; this means that all the field is contributing, if $p=1$, only the active part of turbulence is considered because $\Delta_1$ is equal to the classical fractal dimension $D_f\approx2.36$ (as in the $\beta$ model [3]) and, finally, if $p$ tends to infinity, then it gives the dimension $\Delta_\infty=1$ of the most intermittent structures which display filamentary forms. It suggests interpreting the order $p$ as a thresholding quantity on the fluctuation of the rate of energy dissipation. In fact, it is easily observed that when $p$ increases, the largest fluctuations of $\varepsilon_r$ take on more importance in the final value of the moment $\langle\varepsilon_r^p\rangle$. We thus consider that $\Delta_p$ is the fractal dimension of the support contributing to the moment $\langle\varepsilon_r^p\rangle$; in a recent paper, these supports have been called the *fractal skins* of fully developed turbulence [14].

The calculation of parameter $\beta$ allows now a verification of this internal symmetry in the multifractal spectrum and, more particularly, of relations $\Delta(\alpha)=\lambda-\alpha/\ln\beta$, (2) and (3) (with the calculated value of $\beta$ ($\approx$0.68), we have $\lambda\approx0.065$ and $-1/\ln\beta\approx2.59$). For this, we used the measurements of dimensions $F(\alpha)$ characterizing energy dissipation field, obtained by Arneodo *et al.* [15] through a turbulent velocity signal obtained at Modane. From these values, we calculated the $\Delta(\alpha)$ dimensions (only for $p\geqslant0$). Both quantities are represented in Fig. 2. We observe that the function $\Delta(\alpha)$ is close from a linear variation [the linear interpolation gives $\Delta(\alpha)\approx0.02+2.63\alpha$, in very good agreement with theoretical

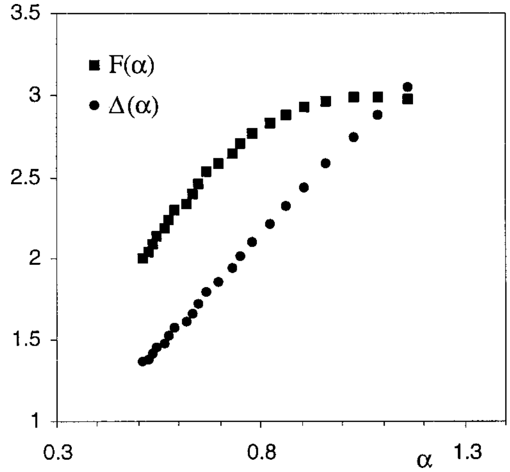


FIG. 2. Measurements of $F(\alpha)$ from Ref. [15] (for $p\geqslant 0$) and its corresponding $\Delta(\alpha)$ dimensions as a function of $\alpha$.

approach] which immediately implies relations (2) and (3). One can observe that, λ being relatively small, relation (2) can be approximated by $\Delta(\alpha_1)/\Delta(\alpha_2)=\alpha_1/\alpha_2$.

Let us come back to the interpretation of $\Delta_p$ dimensions. So, if fully developed turbulence displays a set of fractal structures $\Omega_p$ of dimension $\Delta_p$ linked to each other by the recursive relation $(\Delta_{p+1}-\Delta_\infty)/(\Delta_p-\Delta_\infty)=\beta$, it should be possible to start from this point of view in order to determine its scaling properties. Let us calculate the moments $\langle\varepsilon_r^p\rangle$. For this, we have to introduce some adequate quantities. The fractal structures $\Omega_p$ are characterized by space-filling properties defined by the volume-fraction $f_p(r)=V_p(r)/V_T$. $V_p(r)$ is the volume occupied by $\Omega_p$ at the scale $r$ and $V_T=r_0^d$ is the total volume of the field, where $r_0$ is the integral scale. Since $\Omega_p$ is fractal, it can be written $V_p(r)=(r_0/r)^{\Delta_p}r^d$ and thus $f_p(r)=(r/r_0)^{d-\Delta_p}$. To be able to calculate $\langle\varepsilon_r^p\rangle$, we need a quantity experimentally accessible. Let $P$ be the rate of energy transferred which, in equilibrium conditions of energy injection and energy dissipation, is assumed to be constant: $P\approx(\rho r_0^d)(v_0^3)/r_0$, where $v_0=\langle\delta V_{r0}^2\rangle^{1/2}$ is the typical velocity associated to the integral scale and $\rho$ the fluid density. The rate of energy dissipated per unit mass noted $\overline{\varepsilon_r}$ is calculated through $\overline{\varepsilon_r}=P/m(r)$, where *m(r)* is the mass of the fluid contained in the active part (which is assumed to be fractal of dimension $D_f$) of the field at the scale $r$. The quantity *m(r)* is expressed by $m(r)=\rho r_0^d(r/r_0)^{d-D_f}$. Then, we obtain the scaling relation $\overline{\varepsilon_r}\sim r^{D_f-d}$. We insist on the fact that the quantity $\overline{\varepsilon_r}$ is really different from $\langle\varepsilon_r\rangle$. The first one is an average only over active parts and the second one (which is classically used) is an average over all the field integrating active and nonactive parts. In the calculation of $\langle\varepsilon_r^p\rangle$, there are two contributions: $\overline{\varepsilon_r^p}$ due to the active part and the intermittent factor $f_p(r)$ quantifying the space filling properties of the active part. It gives $\langle\varepsilon_r^p\rangle\sim\overline{\varepsilon_r^p}f_p(r)$. If we assume the existence of scaling laws $\langle\varepsilon_r^p\rangle\sim r^{\tau_p}$, then it leads to $\tau_p=(D_f-d)p+d-\Delta_p$. Since $D_f-d=(d-\Delta_\infty)(\beta-1)$ and $\Delta_p=\Delta_\infty+2\beta^p$, then we obtain $\tau_p=2(\beta-1)p+2(1-\beta^p)$ taking $d=3$ and $\Delta_\infty=1$. We recover the She-Lévêque expression. Nevertheless, our derivation is different from the SL one because it is based only on geometrical arguments. Moreover, using the relative moment $\varepsilon_r^{(p)}=\langle\varepsilon_r^{p+1}\rangle/\langle\varepsilon_r^p\rangle$, it can be easily shown that $\varepsilon_r^{(p+1)}/\overline{\varepsilon_r}\sim(\varepsilon_r^{(p)}/\overline{\varepsilon_r})^\beta$. So, we find a similar relation to the SL fundamental hypothesis [7] where $\overline{\varepsilon_r}$ should be associated with $\varepsilon_r^{(\infty)}$. However, the quantity $\overline{\varepsilon_r}$ does not characterize scaling properties of filaments but mean-field properties of energy dissipation. For this reason, in our description, intermittency is not due to filaments but to this specific hierarchy, beginning with $\Omega_1$ (associated with $\overline{\varepsilon_r}$) and linking all the fractal skins $\Omega_p$ to each other.

As a conclusion, it has been shown that statistics of fully developed turbulence reveals the existence of an internal symmetry in its multifractal spectrum calculated assuming a log-Poisson distribution through She-Lévêque's model [7]. The multifractal dimensions $F(\alpha)$ can be simply written under the form $F(\alpha)=1+\alpha^*-\alpha^*\ln(\alpha^*/2)$, with $\alpha^*=(2\beta-1-\alpha)/\ln\beta=2\beta^p$ where $\beta=[(1+3/\sqrt{8})^{1/3}+(1-3/\sqrt{8})^{1/3}]^3\approx 0.68$. Introducing the dimensions $\Delta_p=F(\alpha)+\alpha^*\ln(\alpha^*/2)$, this leads to the recursive relation $\beta=(\Delta_{p+1}-\Delta_\infty)/(\Delta_p-\Delta_\infty)$ with $\Delta_\infty=1$. The dimension $\Delta_p$ is interpreted as the spatial support of the main contribution to the moment $\langle\varepsilon_r^p\rangle$ based on energy dissipation. This hierarchical structure of fractal skins between all the dimensions $\Delta_p$, directly linked to the multifractal spectrum, reduces considerably the number of parameters needed to characterize the statistical geometry of intermittency. We think that this result could be very useful to describe some other systems displaying multifractal or intermittency statistics such as magneto-hydrodynamic turbulence [16] or diffusion-limited aggregation [17,18]. Of course, this vision requires more developments, and we expect to publish our results very soon.

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